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The Oscillation of a Forced Equation Implies the Oscillation of the Unforced Equation—Small Forcings

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In the present paper we study the following types of equations:

$$x^{(n)} + H(t, x) = Q(t), \qquad n \geqslant 2, \tag{I}$$

$$x^{(n)} + P(t)\, x^{(n-2)} + H(t, x) = 0, \qquad n \geqslant 4, \tag{II}$$

where n is even, H is increasing in its second variable, and $uH(t, u) > 0$ for $u \neq 0$. We first show (Theorem 2.1) that if the homogeneous equation

$$x^{(n)} + H(t, x) = 0 \tag{I_H}$$

has a positive solution, then (I) also has a positive solution provided the function Q is "small" in a certain sense. *An interesting consequence of this result* (*Corollary* 2.1), *is the fact that if Q is "small" and oscillatory, then* (I) *is oscillatory "if and only if"* (I_H) *is oscillatory*. The sufficiency part of this result is contained in Theorem 3.4 of [2]. Next, an oscillation result is given (Theorem 3.1) for the solutions of (II), according to which the oscillation of (I_H) is maintained by the (generalized) damping. In Theorem 3.2, a qualitative property is given for a class of nonoscillatory solutions of (II) characterized by their initial conditions. This theorem is then used in connection with Lemma A in the preliminaries to show that certain solutions with such initial conditions must be oscillatory. A result for n odd is given at the end of Section 3.

Our results in Section 3, complement those of the author in [3] and have points of contact with certain results of Lazer [6] and Heidel [1] who considered, among other things, special cases of Eq. (II) with $n = 3$. The reader is referred to the survey article [4] for an account of results pertaining to the oscillation of forced and perturbed even order equations.

1. PRELIMINARIES

In whct follows, $R = (-\infty, \infty)$ and $R_+ = [0, \infty)$. The function H in (I), (II) will always be defined on $R_+ \times R$ with values in R. Moreover, $H(t, u)$ is

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increasing in u and such that $uH(t, u) > 0$ for $u \neq 0$. The functions P, Q are defined on R_+ with values in R. All functions considered in this paper will be assumed continuous on their domains. Now consider the equation

$$x^{(n)} + F(t, x, x', \ldots, x^{(n-1)}) = 0 \tag{1.1}$$

with $F: R_+ \times R^n \to R$. By a solution of (1.1) we mean here any function $x \in C^n[t_x, \infty)$ which satisfies (1.1) on $[t_x, \infty)$. The number $t_x \geqslant 0$ will depend in general on $x(t)$. A solution of (1.1) is "oscillatory" if it has an unbounded set of zeros in R_+. If all solutions of (1.1) are oscillatory, then (1.1) is said to be "oscillatory."

Now let $x(t)$ solve (1.1). Then x belongs to the class $B(T, k)$ if there exists an odd number k ($1 \leqslant k \leqslant n - 1$) and $T \in R_+$ such that, for every $t \geqslant T$,

$$\begin{aligned} (-1)^i x^{(i)}(t) &< 0, \qquad i = k+1, k+2, \ldots, n, \\ x^{(i)}(t) &> 0, \qquad i = 0, 1, 2, \ldots, k. \end{aligned}$$

The proof of the following lemma follows as in Lemma 1.2 in [2].

LEMMA A. *Suppose that $z \in B(T, k)$ for some $(T, k) \in R_+ \times \{1, 2, \ldots, n-1\}$ and*

$$z^{(n)}(t) + G(t, z(t)) \leqslant 0, \qquad t \geqslant T,$$

where $G: R_+ \to R$ is such that $G(t, u)$ is increasing in u and has the sign of u. Then for every x_0 with $0 < x_0 \leqslant z(T)$, there exists a solution $x(t)$ of the equation

$$x^{(n)} + G(t, x) = 0, \qquad t \geqslant T$$

such that $x(t_0) = x_0$ and $x \in B(T, k)$.

LEMMA B. *Assume that n is an even integer greater than 2. Let the function $x: [a, \infty) \to R$, $a \geqslant 0$ be such that $x^{(i)}(t) > 0$ for $i = 1, 2, \ldots, n-1$ and $t \geqslant a$, and $x^{(n)}(t) < 0$ for $t \geqslant a$. Then there exists $T \geqslant a$ such that*

$$x(t) \geqslant \mu t^{n-2} x^{(n-2)}(t)$$

for every $t \geqslant T$, where μ is a constant independent of $x(t)$.

For a proof of the above lemma the reader is referred to Kiguradge [5].

2. THE EQUATION (I)

THEOREM 2.1. *Consider Eq.* (I) *and assume the existence of a function $S: R_+ \to R$ such that $S^{(n)}(t) \equiv Q(t)$, $t \in R_+$, and $\lim_{t\to\infty} S(t) = 0$. Assume further* *that* (I_H) *has a solution* $x \in B(T, k)$ *for some* $(T, k) \in R_+ \times \{1, 2,..., n-1\}$. *Then given* ϵ *with* $0 < 2\epsilon < x(T)$, *let* $T_1 \geqslant T$ *be such that* $|S(t)| < \epsilon$ *for every* $t \geqslant T_1$. *Then there exists a solution* $z(t)$ *of* (I) *such that* $0 < z(T_1) < x(T_1)$ *and* $z(t) > 0$ *for* $t \geqslant T_1$.

Proof. Note first that since $x'(t) > 0$ for $t \geqslant T_1$, $x(T_1) - \epsilon \geqslant x(T) - \epsilon > 0$. Now consider the function $u(t) = x(t) + S(t)$. Then

$$u^{(n)}(t) + H(t, u(t) - S(t)) = Q(t), \qquad t \geqslant T_1.$$

Now since H is increasing, $H(t, u(t) - S(t)) \geqslant H(t, u(t) - \epsilon) > 0$ for $t \geqslant T_1$ because H has the sign of its second variable and $u(t) - \epsilon = x(t) + S(t) - \epsilon \geqslant x(t) - 2\epsilon \geqslant x(T_1) - 2\epsilon > 0$. Thus,

$$u^{(n)}(t) + H(t, u(t) - \epsilon) \leqslant Q(t), \qquad t \geqslant T_1.$$

Now consider the transformation $v(t) = u(t) - S(t) - \epsilon$, $t \geqslant T_1$. Then

$$v^{(n)}(t) + H(t, v(t) + S(t)) \leqslant 0, \qquad t \geqslant T_1$$

with $v(t) + S(t) = u(t) - \epsilon \geqslant x(T_1) - 2\epsilon > 0$.

Now, essentially Lemma A implies the existence of a solution $w(t) > 0$ $(0 < w(T_1) \leqslant v(T_1) = x(T_1) - \epsilon)$ for the equation

$$w^{(n)} + H(t, w + S(t)) = 0, \qquad t \geqslant T_1.$$

Letting $z(t) = w(t) + S(t)$, $t \geqslant T_1$ we obtain the desired solution of (I) with $z(T_1) = w(T_1) + S(T_1) < x(T_1) - \epsilon + \epsilon = x(T_1)$.

COROLLARY 2.1. *Consider Eq.* (I) *and let the function* $S(t)$ *be as in Theorem* 2.1 *and oscillatory.*

Then (I) *is oscillatory if and only if* (I_H) *is oscillatory.*

Proof. If (I_H) is oscillatory, then (I) is also oscillatory by Theorem 3.4 in [2]. Now let (I) be oscillaotory and assume that (I_H) is nonoscillatory. Then there exists at least one nonoscillatory solhtion of (I_H). Let this solution be $x(t) > 0$ for all large t. Then it is easy to see (cf. also Lemma 2.1 in [2]) that $x \in B(T, k)$ for some $(T, k) \in R_+ \times \{1, 2,..., n-1\}$. Theorem 2.1 implies now that (I) has a positive solution, a contradiction. Now let $x(t) < 0$ for all large t. Then the function $u(t) = -x(t)$ satisfies the equation

$$u^{(n)} - H(t, -u) = 0. \tag{2.1}$$

Now the function $G(t, u) = -H(t, -u)$ is increasing in u and $uG(t, u) > 0$

for $u \neq 0$. Thus, applying Theorem 2.1 to Eq. (2.1) we easily obtain a positive solution $v(t)$ to the equation

$$v^{(n)} - H(t, -v) = -Q(t). \tag{2.2}$$

Letting $z(t) = -v(t)$ we find a negative solution of (I), a contradiction. This completes the proof of the corollary.

It would be quite interesting to know whether the above corollary is true for periodic-like forcings of the form (ii) in Theorem 2.4 of [2]. We should remark here that Theorem 2.1 and Corollary 2.1 actually hold for bounded solutions if the assumptions concern only the bounded solutions of the respective equations.

3. The Equation (II)

THEOREM 3.1. *Consider Eq.* (II) *with* $P(t) \geqslant 0$, $[t^2P(t)]' \leqslant 0$ *for* $t \in R_+$ *and assume that the equation* $u'' + P(t)\, u = 0$ *is nonoscillatory. Moreover, let*

$$\int_0^\infty t^2H(t, \pm k)\, dt = +\infty$$

hold for any $k > 0$. *Then if* (I_H) *is oscillatory,* (II) *is also oscillatory.*

Proof. Let $x(t)$ be a solution of (II) such that $x(t) > 0$ for all large t. Then Theorem B in [3] imples that $x^{(n-2)}(t) > 0$ or $x^{(n-2)}(t) < 0$ for all large t. Let the first inequality hold. Then from (II) we obtain

$$x^{(n)}(t) + H(t, x(t)) \leqslant x^{(n)}(t) + P(t)\, x^{(n-2)}(t) + H(t, x(t)) = 0$$

for all large t. Using Lemma A now we deduce that the Eq. (I_H) also has a solution which is eventually positive. Consequently, we arrive at a contradiction. Thus, $x^{(n-2)}(t) < 0$ eventually. Since n is even, we must also have $x^{(n-3)}(t) > 0$ for all large t; otherwise, two consecutive derivatives which are eventually negative will imply $x(t) \to -\infty$ as $t \to +\infty$, a contradiction. Thus, there exists $t_1 \geqslant 0$ suchtthct $x(t) > 0$, $x^{(n-3)}(t) > 0$ and $x^{(n-2)}(t) < 0$ for every $t \geqslant t_1$. Again, since n is even, we must have $x'(t) > 0$ for all large t, say for $t \geqslant t_1$. Now we multiply (II) by t^2 and we integrate from t_1 to t to obtain

$$t^2x^{(n-1)}(t)\Big|_{t_1}^{t} - 2\int_{t_1}^{t} sx^{(n-1)}(s)\, ds + s^2P(s)\, x^{(n-3)}(s)\Big|_{t_1}^{t} - \int_{t_1}^{t} [s^2P(s)]'\, x^{(n-3)}(s)\, ds$$

$$= -\int_{t_1}^{t} s^2H(s, x(s))\, ds \leqslant -\int_{t_1}^{t} s^2H(s, x(t_1))\, ds. \tag{3.1}$$

Since the function $t^2P(t)$ is nonnegative and decreasing, and since $x^{(n-3)}(t)$ has the same property, the limit $\lim_{t\to\infty} t^2P(t)\, x^{(n-3)}(t)$ exists and is a finite number. Thus, taking limits as $t \to +\infty$ in (3.1), and taking into account the integral condition on H, we obtain

$$\lim_{t\to\infty} \left[t^2x^{(n-1)}(t) - 2\int_{t_1}^{t} sx^{(n-1)}(s)\, ds\right] = -\infty.$$

Now we apply Lemma 1 of Staikos and Sficas [8] to obtain that

$$\begin{aligned}\lim_{t\to\infty} \int_{t_1}^{t} sx^{(n-1)}(s)\, ds &= \lim_{t\to\infty}\{[tx^{(n-2)}(t)]_{t_1}^{t} - [x^{(n-3)}(t) - x^{(n-3)}(t_1)]\} \\ &= \pm\infty.\end{aligned} \tag{3.2}$$

Obviously, since $x^{(n-2)}(t) < 0$ and $x^{(n-3)}(t) > 0$, the above limit must equal $-\infty$.

Now since $\lim_{t\to\infty} x^{(n-3)}(t) = \lambda$ $(0 \leqslant \lambda < +\infty)$, we must have $\lim_{t\to\infty} tx^{(n-2)}(t) = -\infty$. Thus, there is a $t_2 \geqslant t_1$ and $k < 0$ such that $x^{(n-2)}(t) \leqslant k/t$ for $t \geqslant t_2$. Integrating this last inequality from t_2 to $t \geqslant t_2$ we obtain that $\lim_{t\to\infty} x^{(n-3)}(t) = -\infty$, a contradiction. This completes the proof of the theorem, because an almost identical argument can be used in the case of an assumed negative $x(t)$.

THEOREM 3.2. *Consider Eq.* (II) *with $P(t) \geqslant 0$ and $P'(t) \leqslant 0$ in R_+. Moreover, let the equation $u'' + P(t)\, u = 0$ be nonoscillatory and let $x(t)$, $t \in [t_1, \infty)$, be an eventually positive solution of* (II). *Let*

$$F(x(t)) = [x^{(n-2)}(t)]^2 - 2x^{(n-3)}(t)\, x^{(n-1)}(t) - P(t)\, [x^{(n-3)}(t)]^2 \tag{3.3}$$

for $t \geqslant t_1$. Then one of the following holds. (i) *There exists $t_2 \geqslant t_1$ such that $x(t) > 0$, $x^{(n-3)}(t) > 0$, $x^{(n-2)}(t) > 0$, $x^{(n-1)}(t) > 0$, $x^{(n)}(t) < 0$ for every $t \geqslant t_2$.* (ii) *$F(x(t)) < 0$ for all large t.*

Proof. Since $x(t)$ is eventually positive, Theorem B in [3] implies that $x^{(n-2)}(t) > 0$ or $x^{(n-2)}(t) < 0$ for all large t. In either case, we must have $x^{(n-3)}(t) > 0$ eventually. In fact, if $x^{(n-2)}(t) > 0$ eventually, our assertion follows from $x^{(n)}(t) < 0$, $x^{(n-1)}(t) > 0$ eventually. If $x^{(n-2)}(t) < 0$ then $x^{(n-3)}(t)$ cannot be negative for all large t because this would contradict the positiveness of $x(t)$. Now let $x^{(n-3)}(t) > 0$ for $t > t_2 \geqslant t_1$. If $t_2 > t_1$, then there may be a number $t_3 \in [t_1, t_2)$ such that $x^{(n-3)}(t_3) = 0$ and $x^{(n-3)}(t) > 0$ for $t > t_3$. At this point t_3 we must have $F(x(t_3)) = [x^{(n-2)}(t_3)]^2 \geqslant 0$. Thus, we may always choose t_1 so that $F(x(t_1)) \geqslant 0$ and $x^{(n-3)}(t) > 0$ for $t > t_1$. Let $x(t_1) > 0$ for $t \geqslant t_1$. Then differentiating $F(x(t))$ on $[t_1, \infty)$ and then integrating from t_1 to $t \geqslant t_1$ we obtain

$$F(x(t)) = F(x(t_1)) + \int_{t_1}^{t} [2H(s, x(s)) - P'(s)\, x^{(n-3)}(s)]\, x^{(n-3)}(s)\, ds.$$

Consequently, $F(x(t))$ is a strictly increasing function on $[t_1, \infty)$ with $F(x(t_1)) \geqslant 0$. Now assume that $x^{(n-2)}(a) = 0$ for some $a \in (t_1, \infty)$. Then since

$$F(x(a)) = -2x^{(n-3)}(a)\, x^{(n-1)}(a) - P(a)\, [x^{(n-3)}(a)]^2 > 0,$$

we must have $x^{(n-1)}(a) < 0$. Consequently, $x^{(n-2)}(t)$ cannot have more than one zero on (t_1, ∞). Now assume that $x^{(n-2)}(t) < 0$ eventually. Then integrating (II) from t_1 to $t \geqslant t_1$ we obtain

$$\begin{aligned} x^{(n-1)}(t) - x^{(n-1)}(t_1) + P(t)\, x^{(n-3)}(t) - P(t_1)\, x^{(n-3)}(t) \\ - \int_{t_1}^{t} P'(s)\, x^{(n-3)}(s)\, ds + \int_{t_1}^{t} H(s, x(s))\, ds = 0. \end{aligned} \tag{3.4}$$

Now $\lim_{t\to\infty} P(t) = 0$, otherwise $u'' + P(t)\, u = 0$ would be oscillatory by Wintner's criterion [9]. Thus, since $x^{(n-3)}(t)$ is bounded, $\lim_{t\to\infty} P(t)\, x^{(n-3)}(t) = 0$. Also

$$- \int_{t_1}^{\infty} P'(s)\, x^{(n-3)}(s)\, ds + \int_{t_1}^{\infty} H(s, x(s))\, ds = \lambda < +\infty,$$

otherwise $x^{(n-1)}(t) \to -\infty$ as $t \to +\infty$, a contradiction to the positiveness of $x(t)$. Consequently, $\lim_{t\to\infty} x^{(n-1)}(t)$ exists and equals zero. If this limit were positive, then $\lim_{t\to\infty} x^{(n-2)}(t) = +\infty$, a contradiction. If it were negative, then $\lim_{t\to\infty} x(t) = -\infty$, a contradiction again. Consequently, taking limits in (3.3) we obtain

$$\lim_{t\to\infty} F(x(t)) = \lim_{t\to\infty} [x^{(n-2)}(t)]^2 = \lambda.$$

Now λ cannot be positive, because if it were, then $\lim_{t\to\infty} x^{(n-2)}(t) = -\lambda^{1/2}$, a contradiction to the positiveness of $x(t)$. Thus, $\lim_{t\to\infty} F(x(t)) = 0$, a contradiction again because $F(x(t))$ is strictly increasing in $[t_1, \infty)$. Consequently, $x^{(n-2)}(t) > 0$ for all large t. Then (II) implies that $x^{(n)}(t) < 0$ eventually, which in turn yields $x^{(n-1)}(t) > 0$ eventually. This completes the proof.

The above theorem has been shown for $n = 3$ by Lazer [6], but our proof is different. In the following theorem conditions are provided to ensure that (II) does not have any solutions $x(t)$ with $x(t)\, x^{(n-2)}(t) \geqslant 0$ eventually. Thus, all nonoscillatory solutions must be such that $F(x(t)) < 0$ for all large t. The constant μ below is as in Lemma B.

THEOREM 3.3. *Let the assumptions on P of Theorem 3.2 be satisfied. Moreover, assume that for some constant k such that $0 < k \leqslant \mu$ the equation $u'' + P(t)\, u + H(t, kt^{n-2}u) = 0$ is oscillatory. Then every solution $x(t)$, $t \in [t_1, \infty)$, $t_1 \geqslant 0$, is oscillatory or such that $F(x(t)) < 0$ for all large t.*

Proof. Let $x(t)$, $t \geqslant t_1$ be a solution of (I) such that $F(x(t_2)) \geqslant 0$ and $x(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. Then Theorem 3.2 implies the existence of some $t_3 \geqslant t_2$ such that $x^{(n)}(t) < 0$, $x^{(n-1)}(t) > 0$, $x^{(n-2)}(t) > 0$, and $x^{(n-3)}(t) > 0$ for all $t \geqslant t_3$. Now we can apply Lemma B, according to which $x(t) \geqslant \mu t^{n-2} x^{(n-2)}(t)$ for every $t \geqslant t_4$ with $t_4 \geqslant t_3$. Thus, from (II) we obtain

$$x^{(n)}(t) + P(t)\, x^{(n-2)}(t) + H(t, \mu t^{n-2} x^{(n-2)}(t)) \leqslant 0, \qquad t \geqslant t_4 .$$

Now we let $u(t) = x^{(n-2)}(t)$, $t \geqslant t_4$. Then the function $u(t)$ satisfies the inequalities

$$u'' + P(t)\, u + H(t, kt^{n-2}u) \leqslant u'' + P(t)\, u + H(t, \mu t^{n-2}u) \leqslant 0.$$

By Lemma A, the equation

$$u'' + P(t)\, u + H(t, kt^{n-2}u) = 0$$

must also have a positive solution for $t \geqslant t_5 \geqslant t_4$, a contradiction to our assumption. Assume now that $x(t) < 0$ eventually. Then $u(t) = -x(t) > 0$ eventually and satisfies the equation

$$u^{(n)} + P(t)\, u^{(n-2)} - H(t, -u) = 0$$

for all large t. Since $v(-H(t, -v)) > 0$ for $v \neq 0$, $-H(t, -v)$ is increasing in v assuming $F(u(t_4)) \geqslant 0$ and $u(t_2) > 0$, $t \geqslant t_2$, Theorem 3.2 implies again that $u^{(n-1)}(t) > 0$, $u^{(n-2)}(t) > 0$, $u^{(n-3)}(t) > 0$, and $u^{(n)}(t) < 0$ eventually. Proceeding as above, we finally obtain a positive solution $z(t)$ of the equation

$$u'' + P(t)\, u - H(t, -kt^{n-2}u) = 0.$$

Letting $w(t) = -z(t)$ we obtain the desired contradiction. This completes the proof of the theorem.

The above theorem was proved by Lazer [6] in the llnear case and for $n = 3$. Lazer applied a transformation method involving Sturm's comparison theorem. For special cases of the function H and for $n = 3$ the reader is referred also to Heidel [1; Theorem 3.2, Lemma 3.3, Corollary 3.4].

From Theorem 3.3 we can easily obtain now the following:

COROLLARY 3.1. *Let the functions P, H satisfy the assumptions of Theorem* 3.3; *then a solution $x(t)$, $t \in [t_1, \infty)$, $t_1 \geqslant 0$ of* (II) *is nonoscillatory if and only if $F(x(t)) < 0$ for all large t.*

Proof. If $F(x(t)) < 0$ for all large t, $x(t)$ cannot be oscillatory because if this were the case, then at each zero $\bar{t}$ of $x^{(n-3)}(t)$—which would also be oscillatory by Rolle's theorem—we would have $F(x(\bar{t})) \geqslant 0$, a contradiction. On the other hand, if $x(t)$ is nonoscillatory and $F(x(t_m)) \geqslant 0$, with $t_{m\to\infty}$ as $m \to \infty$ then we obtain contradiction from Theorem 3.3. This completes the proof.

The functional F encountered above seems to be a valuable tool for equations of the type (II), especially when $P(t) \not\equiv 0$. To illustrate its usefulness further we present another theorem, which concerns the case of n odd.

THEOREM 3.4. *In Eq.* (II) *let* $n \geqslant 3$ *be odd. Moreover let* $P(t) \geqslant 0$, $P'(t) \leqslant 0$ *and such that* $u'' + P(t)u = 0$ *is nonoscillatory, while* $u'' + P(t)\,u + H(t, \mu t^{n-2}u) = 0$ *is oscillatory, where* μ *is as in Lemma* B. *Assume further that every solution of* (II) *is either oscillatory or tending monotonically to zero as* $t \to +\infty$. *Then every solution of* (II) *with a zero is oscillatory.*

Proof. Let $x(t)$, $t > t_1$, $t_1 \geqslant 0$ be a nonoscillatory solution of (II) with $x(t_1) = 0$. Then there is a last zero of $x(t)$, say, $t_2 \geqslant t_1$. For $t > t_2$, $x(t) \gtrless 0$. Let $x(t) > 0$, $t \geqslant t_2$. Then since $x(t) \to 0$ as $t \to +\infty$, we must have $(-1)^i x^{(i)}(t) \geqslant 0$ for $i = 1, 2,\ldots, n-2$ and all large t. Assume now that $x^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Then $x^{(n-4)}(t) < 0$ for $t \geqslant t_2$. In fact, if this were not true, then there would exist $t_3 \geqslant t_2$ such that $x^{(n-4)}(t_3) = 0$, $x^{(n-4)}(t) < 0$ for $t \geqslant t_3$. However, since $x^{(n-4)}(t)$ is strictly increasing we must have $x^{(n-4)}(t) > 0$ for $t \geqslant t_3$, a contradiction. This way we can show that none of the derivatives $x'(t)$, $x''(t),\ldots, x^{(n-3)}(t)$ has a zero in $[t_2, +\infty)$. Thus, $x'(t) < 0$ for $t \geqslant t_2$, a contradiction because $x(t_2) = 0$ and $x(t) > 0$ for $t > t_2$. It follows that $x^{(n-3)}(t)$ lust have a zero $\bar{t} \geqslant t_2$. Consequently, $F(x(\bar{t})) = [x^{(n-2)}(\bar{t})]^2 \geqslant 0$, and this in connection with the proof of Theorem 3.3 (for n odd) implies that $x(t)$ must be oscillatory, a contradiction. If $x(t) < 0$ for $t \geqslant t_2$, then $-x(t)$ satisfies the properties of $x(t)$ above with $F(-x(\bar{t})) \geqslant 0$ and again Theorem 3.3 implies a contradiction. This completes the proof.

This theorem is actually an improvement of a result of the author in [3] (see remarks before Theorem 5 therein). It is also intimately related to, and extends the result in, the remark following Corollary 3.4 of Heidel's paper [1].

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